

Birinchi va ikkinchi ajoyib limitlar

Matematik analizda muhim ahamiyatga ega bo'lgan va *ajoyib (muhim) limitlar* deb ataluvchi ikkita limitni keltiramiz.

9.5.1. Birinchi ajoyib limit

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (9.5.1)$$

bo'lishini ko'rsataylik. Buning uchun $f(x) = \frac{\sin x}{x}$ funksiyani qaraymiz. Bu funksiya 0 nuqtada aniqlanmagan, ammo, uning yaqin atrofida mavjud va juft funksiyadir. Demak, $x > 0$ holni qarash yetarlidir. Undan tashqari, $x \rightarrow 0$ ekanligidan $x \in (0; \frac{\pi}{2})$ deb olsak bo'ladi.

Dastlab, $\lim_{x \rightarrow 0} \sin x = 0$, $\lim_{x \rightarrow 0} \cos x = 1$ ekanligini isbotlaylik. Buning uchun birlik doira olib (9.5.1-rasm), x markaziy burchak $(0; \frac{\pi}{2})$ oraliqda bo'lsin deylik. U vaqtda AB yoy uzunligi radianlarda ifodalangan x markaziy burchak o'lchoviga tengdir. 9.5.1-rasmdan ko'rinadiki, BD kesma $\sin x$ ga, AC esa $\tan x$ ga teng bo'lib, $0 < BD < AB < AC$ o'rinlidir. Bundan

$$0 < \sin x < x$$

kelib chiqadi. Bu tengsizliklardan va limitning mavjudligi haqidagi ikkinchi teoremdan $\lim_{x \rightarrow +0} \sin x = 0$ ni olamiz. Funksiyaning toqlik xossasiga ko'ra $\lim_{x \rightarrow -0} \sin x = 0$ va olinganlar asosida $\lim_{x \rightarrow 0} \sin x = 0$ ga ega bo'lamiz.

Endi,

$$\cos x = 1 - 2 \sin^2 \frac{x}{2}$$

ekanligini e'tiborga olsak,

$$\lim_{x \rightarrow 0} \cos x = \lim_{x \rightarrow 0} \left(1 - 2 \cdot \sin^2 \frac{x}{2} \right) = 1 - 2 \cdot 0^2 = 1$$

kelib chiqadi.

9.5.1- rasmdan $\triangle OAB$, $\triangle OAC$ va OAB sektor yuzlari uchun

$$S_{\triangle OAB} < S_{\text{sektor } OAB} < S_{\triangle OAC}$$

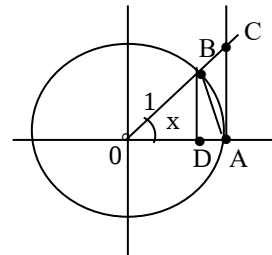
o'rinli ekanligini ko'ramiz.

$$S_{\triangle OAB} = \frac{1}{2} OA \cdot OB \sin x = \frac{1}{2} \sin x, \quad S_{\text{sektor } OAB} = \frac{1}{2} OA^2 x = \frac{1}{2} x, \quad S_{\triangle OAC} = \frac{1}{2} OA \cdot AC = \frac{1}{2} \tan x.$$

Yuqoridagi tengsizliklarda bularni va $x \in (0; \frac{\pi}{2})$ ni hisobga olsak,

$$\frac{\sin x}{2} < \frac{x}{2} < \frac{\tan x}{2}$$

ni, bundan



9.5.1- rasm

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

ni olamiz va, nihoyat,

$$\cos x < \frac{\sin x}{x} < 1$$

ga ega bo‘lamiz. Oxirgida $\lim_{x \rightarrow 0} \cos x = 1$ ni hisobga olib $x \rightarrow +0$ dagi limitga o‘tsak,

$$\lim_{x \rightarrow +0} \frac{\sin x}{x} = 1$$

kelib chiqadi. Funksiyaning juftligidan

$$\lim_{x \rightarrow -0} \frac{\sin x}{x} = 1$$

ekanligi ham kelib chiqadi. Bulardan *birinchi ajoyib limit* deb ataluvchi (9.5.1) ning o‘rinli ekanligiga kelamiz.

Misollar.

$$1) \lim_{x \rightarrow 0} \frac{\sin kx}{x} = \lim_{x \rightarrow 0} \left(k \cdot \frac{\sin kx}{kx} \right) = k \cdot \lim_{x \rightarrow 0} \frac{\sin kx}{kx} = k \cdot 1 = k;$$

$$2) \lim_{x \rightarrow 0} \frac{\operatorname{tg} x}{x} = \lim_{x \rightarrow 0} \left(\frac{1}{\cos x} \cdot \frac{\sin x}{x} \right) = \frac{1}{1} \cdot 1 = 1.$$

9.5.2. Ikkinchi ajoyib limit

Avvalo, $\left\{ \left(1 + \frac{1}{n} \right)^n \right\}$ ketma-ketlikning limiti mavjud ekanligini isbotlaymiz.

Umumiy hadlari mos ravishda $x_n = \left(1 + \frac{1}{n} \right)^n$ va $y_n = \left(1 + \frac{1}{n} \right)^{n+1}$ bo‘lgan $\{x_n\}$ va $\{y_n\}$

ketma-ketliklarni olsak, ular orasida $y_n = x_n \left(1 + \frac{1}{n} \right)$ bog‘lanish bordir.

Endi, $\{y_n\}$ ketma-ketlikni tekshiramiz.

$$y_n = \left(1 + \frac{1}{n} \right)^{n+1} > 1^{n+1} = 1$$

ekanligi ravshandir. Demak, $\{y_n\}$ quyidan chegaralangan ketma-ketlik ekan. Bu ketma-ketlikning kamayuvchi ekanligini ko‘rsatishda tubandagi tasdiq qo‘l keladi.

Lemma (Bernulli). $m \in \mathbb{N}$ va $\alpha > -1$ bo‘lganda $(1 + \alpha)^m \geq 1 + m\alpha$ tengsizlik o‘rinlidir.

Isbot. Matematik induksiya uslubidan foydalanamiz:

1) $m = 1 \Rightarrow (1 + \alpha)^1 = 1 + \alpha$ o‘rinli;

2) $m = k$ uchun $(1 + \alpha)^k \geq 1 + k\alpha$ tengsizlik o‘rinli deb faraz qilaylik;

3) U holda, $1 + \alpha > 0$ ni hisobga olsak,

$$\begin{aligned} (1 + \alpha)^{k+1} &= (1 + \alpha)^k \cdot (1 + \alpha) \geq (1 + k\alpha)(1 + \alpha) = 1 + \alpha + k\alpha + k\alpha^2 = \\ &= 1 + (k+1)\alpha + k\alpha^2 \geq 1 + (k+1)\alpha, \end{aligned}$$

ya’ni $m = k + 1$ uchun ham tengsizlik o‘rinli ekanligi kelib chiqadi. Lemma isbotlandi.

Eslatma. Agar $m > 1$ va $\alpha \neq 0, \alpha > -1$ bo‘lsa, $(1 + \alpha)^m > 1 + m\alpha$ o‘rinli bo‘ladi.

Endi, $\frac{y_n}{y_{n+1}}$ nisbatni olsak,

$$\begin{aligned}\frac{y_n}{y_{n+1}} &= \frac{\left(1 + \frac{1}{n}\right)^{n+1}}{\left(1 + \frac{1}{n+1}\right)^{n+2}} = \left(\frac{1 + \frac{1}{n}}{1 + \frac{1}{n+1}}\right)^{n+2} \cdot \frac{1}{1 + \frac{1}{n}} = \left(\frac{(n+1)^2}{n(n+2)}\right)^{n+2} \cdot \frac{n+1}{n} = \\ &= \left(1 + \frac{1}{n(n+2)}\right)^{n+2} \cdot \frac{n}{n+1} > \left(1 + \frac{n+2}{n(n+2)}\right) \cdot \frac{n}{n+1} = \left(1 + \frac{1}{n}\right) \cdot \frac{n}{n+1} = \frac{n+1}{n} \cdot \frac{n}{n+1} = 1,\end{aligned}$$

ya'ni $\frac{y_n}{y_{n+1}} > 1 \Rightarrow y_n > y_{n+1}$. Bundan $\{y_n\}$ kamayuvchi ketma-ketlik ekanligi kelib chiqadi.

Demak, $\{y_n\}$ kamayuvchi quyidan chegaralangan ketma-ketlik sifatida, limitning mavjudligi haqidagi birinchi teorema asosan, chekli limitga egadir.

Endi,

$$x_n = \frac{y_n}{1 + \frac{1}{n}}$$

ekanligidan

$$\lim x_n = \frac{\lim y_n}{\lim \left(1 + \frac{1}{n}\right)} = \frac{\lim y_n}{1} = \lim y_n,$$

ya'ni $\{x_n\}$ ketma-ketlik ham $\{y_n\}$ limitiga teng limitga ega ekanligi kelib chiqadi, uni e bilan belgilanadi va e soni deb ataladi.

Demak, e soni, uning ta'rifi bo'yicha,

$$e = \lim \left(1 + \frac{1}{n}\right)^n$$

dan iboratdir. Bu son irratsional, (2;3) oraliqqa tegishli bo'lib,

$$e = 2,7182818284459045 \dots$$

Matematik analizda,

$$\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e \quad \text{yoki} \quad \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

ikkinchi ajoyib (muhim) limit deb yuritiladi.

Yuqoridagi munosabatlar biridan ikkinchisi cheksiz katta miqdorning teskari qiymati cheksiz kichik ekanligidan bevosita kelib chiqadi.

Endi,

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

ekanligini isbotlaylik. Buning uchun, avvalo, $x \rightarrow +\infty$ bo'lgan holni qaraymiz. Bu holda, $x > 1$ desak, har bir bunday x uchun shunday n natural son topiladiki,

$$n \leq x < n+1$$

o'rinli bo'ladi. Oxiridan

$$1 < 1 + \frac{1}{n+1} < 1 + \frac{1}{x} \leq 1 + \frac{1}{n}$$

ni olamiz. Bundan esa,

$$\left(1 + \frac{1}{n+1}\right)^n < \left(1 + \frac{1}{x}\right)^x < \left(1 + \frac{1}{n}\right)^{n+1}$$

kelib chiqadi.

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n+1}\right)^n = \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{1 + \frac{1}{n+1}} = \frac{e}{1} = e,$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+1} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \cdot \left(1 + \frac{1}{n}\right) = e \cdot 1 = e$$

ekanligidan va $n \rightarrow +\infty \Leftrightarrow x \rightarrow +\infty$ bo'lishini hisobga olsak, limitning mavjudligi haqidagi ikkinchi teorema asosan,

$$\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x = e$$

kelib chiqadi.

Endi, $x \rightarrow -\infty$ bo'lgan holni qarasak va $x = -y$ almashtirish qilsak, quyidagilarni olamiz:

$$\begin{aligned} x \rightarrow -\infty &\Rightarrow y \rightarrow +\infty \Rightarrow (y-1) \rightarrow +\infty \\ \lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x &= \lim_{y \rightarrow +\infty} \left(1 - \frac{1}{y}\right)^{-y} = \lim_{y \rightarrow +\infty} \left(\frac{y-1}{y}\right)^{-y} = \lim_{y \rightarrow +\infty} \left(\frac{y}{y-1}\right)^y = \\ &= \lim_{y \rightarrow +\infty} \left(1 + \frac{1}{y-1}\right)^y = \lim_{y-1 \rightarrow +\infty} \left[\left(1 + \frac{1}{y-1}\right)^{y-1} \cdot \left(1 + \frac{1}{y-1}\right)\right] = e \cdot 1 = e \end{aligned}$$

Shunday qilib, ikkinchi ajoyib limit isbotlandi.

Misol. Ushbu $\lim_{x \rightarrow \infty} \left(1 + \frac{m}{x}\right)^x = e^m$, $m \in \mathbb{N}$ limitning to'g'riligini ko'rsating.

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(1 + \frac{m}{x}\right)^x &= \lim_{x \rightarrow \infty} \left(1 + \frac{m}{x}\right)^{\frac{x}{m} \cdot m} = \lim_{\frac{1}{m}x \rightarrow \infty} \left[\left(1 + \frac{m}{x}\right)^{\frac{x}{m}}\right]^m = \\ &= \lim_{\alpha \rightarrow 0} \left[\left(1 + \frac{1}{\alpha}\right)^{\frac{1}{\alpha}}\right]^m = e^m \quad \left(\frac{m}{x} = \alpha; x \rightarrow \infty \Rightarrow \alpha \rightarrow 0\right). \end{aligned}$$

Eslatma. Matematik analizda asosi e sonidan iborat bo'lgan ko'rsatkichli va logarifmik funksiyalar alohida ahamiyatga egadir. Masalan, e asosli logarifmlar maxsus

$$\log_e a = \ln a$$

belgilashga ega bo'lib, uni *natural logarifm* deb yuritiladi.